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Inversion Number and Collisions in Some Billiard Systems

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Abstract With help of inversion numbers, we obtain sharp upper bounds of the number of collisions in some special billiard systems.

Keywords Inversion number · Billiard · Polyhedral cone · Hard ball · Elastic collision

1 Introduction

This paper is an enhancement of the author's work [2], in which we study a conjecture of T.J. Murphy and E.G.D. Cohen on upper bounds of the number of collisions in a particular kind of hard ball systems on a straight line. We have proven the conjecture under a weaker assumption with help of inversion numbers. The present paper will apply the method to more general systems.

Estimates of the number of collisions in hard ball systems, more generally in semi-dispersing billiards, have been studied for a long time, for instances [1, 3, 5, 6], because of their importance for proper generalization of the Boltzmann equation. For general systems, all known upper bounds of the number of collisions increase exponentially with the number of balls in hard ball systems, or with the number of walls in billiards. The present paper will show some systems with the number of collisions upper bounded by the inversion number of a sequence of length $n+1$ determined by the initial data, where n is the number of walls in billiards, thus we obtain the uniform upper bound $\frac{n(n+1)}{2}$.

Consider a billiard inside an unbounded polyhedron. Suppose the walls of this unbounded polyhedron are indexed as $W_1, \dots, W_n$. We will show that, if the walls W_i and W_j are mutually perpendicular when $|i-j|>1$, and the angle between W_i and W_{i+1} is not less than $\frac{\pi}{3}$ for each i, then the number of collisions of any billiard trajectory in the unbounded polyhedron does not succeed $\frac{n(n+1)}{2}$. Applying this result to a hard ball system on a line with masses $m_0, m_1, \dots, m_n$ successively, we obtain that, if $(1+\frac{m_i}{m_{i-1}})(1+\frac{m_i}{m_{i+1}}) \geq 4$ for each

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i, then the number of collisions of the system does not succeed $\frac{n(n+1)}{2}$. In particular, for a system of hard balls on a (right) half-line, i.e., $m_0 = +\infty$, if $m_1 \geq 3m_2$ and $m_i \geq \sqrt{m_{i-1}m_{i+1}}$ for each $i \geq 2$, then the number of collisions of the system does also not succeed $\frac{n(n+1)}{2}$. As another consequence, if $\frac{m_{2k+1}}{m_{2k}} = \frac{k}{k+1}$ and $\frac{m_{2k}}{m_{2k-1}} = \frac{2k-1}{2k+1}$ for $k \geq 0$, then the number of collisions of the system is not bigger than $\frac{n(n+1)}{2}$. And similar situations hold for hard ball systems on a whole-line (i.e., $m_0 < +\infty$).

We will review in detail the well known fact that any hard ball system is isomorphic to a mathematical billiard over some space in the next section, with emphasis on the case of one-dimensional hard ball systems (including those on a straight line, on a circle, on a half-line, in an interval). It presents necessary preliminaries for the later sections and lays down the style of notation. The isomorphism is important not only for our purposes. Particularly, it immediately follows that hard ball systems are Hamiltonian and it is also useful when questions of the existence of periodic motions are concerned. In Sect. 3, we introduce a set-valued point view of multiple collisions, justifying us to remove the artificial assumption that multiple collisions do not occur for estimates of the number of collisions. And in the last section, we prove the results asserted in the preceding paragraph.

2 From Hard Ball Systems to Billiards

A hard ball system is a system of a finite number of perfectly elastic balls moving freely and undergoing collisions from time to time in a region of m-dimensional Euclidean space. The balls are supposed to be homogeneous with different masses. Their radii may be strictly positive or zero. An equivalent formulation instead of the hypothesis of homogeneity is that the total mass of each ball is concentrated at its geometrical center. In other words, a hard ball system in a space is a system of material particles, in which the interaction of the particles is local, centrally repelling and elastic. The particles do not interact when the distance between them exceeds the sum of the radii of the corresponding balls and centrally repel when the balls collide. It is necessary to take this point of view when we consider hard ball systems in a curved space. The collisions are all perfectly elastic, i.e., they obey the law of conservation of kinetic energy. At the instant of a collision between two balls, the distance between their centers equals the sum of their radii. The interaction is centrally repelling and elastic means that there occurs a redistribution of the components of the velocities in the direction of the line of the centers of the balls in accordance with the laws of conservation of momentum and energy, and the orthogonal components of the velocities remain unchanged. It is in a similar situation when a ball meets the boundary of the region. At this instant, the distance from the center of the ball to the boundary equals the radius of the ball and it is supposed to be achieved only at the tangent point. The tangential component of the velocity of the ball remains unchanged and the normal component of the velocity differs in sign. It is exactly the familiar law: the angle of incidence is equal to the angle of reflection.

A mathematical billiard is a point particle (the billiard ball) moving freely, i.e., in uniform rectilinear motion, in a space (the billiard table, the configuration space), besides reflecting elastically when it hits the boundary of the space. The particle moves along straight lines with constant speed in the interior of the space. And when it reflects off the boundary, the angle of incidence is equal to the angle of reflection. It is well known that the rule of elastic reflection means that the trajectory is a local minimizer of the arc-length functional. Hence a mathematical billiard is just the geodesic flow over the configuration space, whose boundary consists of several smooth components (the walls).

To show arbitrary hard ball system is isomorphic to a billiard over some configuration space, we will study first the simplest case carefully, and draw the general conclusion from it rigorously. This treatment is very concrete and needs few calculations in the same time.

Consider a system of two elastic material points on a straight line. Suppose that the coordinates of the two points are $x_1(t)$ and $x_2(t)$ satisfying $x_1(t) \le x_2(t)$, $\forall t$, and the masses are m_1 and m_2 respectively. Take a scaling:

$$y_i(t) = \sqrt{m_i} x_i(t), \quad i = 1, 2.$$

Then the velocities rescale in the same way:

$$\dot{y}_i(t) = \sqrt{m_i} \dot{x}_i(t), \quad i = 1, 2.$$

Hence the momentum and kinetic energy of the system are, respectively,

$$M = \sqrt{m_1}\dot{y}_1 + \sqrt{m_2}\dot{y}_2 = \left\langle \begin{pmatrix} \sqrt{m_1} \\ \sqrt{m_2} \end{pmatrix}, \begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} \right\rangle \tag{2.1}$$

and

$$E = \frac{1}{2}\left(\dot{y}_1^2 + \dot{y}_2^2\right) = \frac{1}{2}\left\| \begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} \right\|^2. \tag{2.2}$$

They are independent on the time. The configuration space is a half-plane:

$$\left\{ (y_1, y_2) \colon \frac{y_1}{\sqrt{m_1}} - \frac{y_2}{\sqrt{m_2}} \le 0 \right\}.$$

Since $(\sqrt{m_1}, \sqrt{m_2})^T$ is a tangent vector of the boundary of the configuration space, the conservation law of momentum states that the component of the system velocity vector

$$\dot{y} = \begin{pmatrix} \dot{y}_1(t) \\ \dot{y}_2(t) \end{pmatrix}$$

in the tangential direction of the boundary is constant. And the conservation law of kinetic energy states that the length of the system velocity vector is constant. It follows that, if t_0 is an instant of a collision, then the system velocity vectors at the instant before the collision $\dot{y}(t_0-)$ and after the collision $\dot{y}(t_0+)$ are symmetric about the tangential direction. In other words, if we denote σ the orthogonal reflection about the tangent line, then

$$\sigma(\dot{y}(t_0-)) = \dot{y}(t_0+), \qquad \sigma(\dot{y}(t_0+)) = \dot{y}(t_0-). \tag{2.3}$$

Explicitly, let

$$h = \frac{1}{\sqrt{\frac{1}{m_1} + \frac{1}{m_2}}} \begin{pmatrix} -\frac{1}{\sqrt{m_1}} \\ \frac{1}{\sqrt{m_2}} \end{pmatrix}$$

which is the unit inner normal vector of the boundary, then

$$\sigma(v) = v - 2\langle v, h\rangle h, \quad \forall v \in \mathbb{R}^2,$$

written in the matrix form

$$\sigma = I - 2hh^T,$$

where I is the 2×2 identity matrix. In particular, we have

$$\langle \dot{y}(t_0+), h\rangle = -\langle \dot{y}(t_0-), h\rangle, \tag{2.4}$$

which shows that the elastic collision changes the sign of the relative velocity of the two material points. We have also seen the reversibility of elastic collisions.

If we consider two hard balls with radii r_1 and r_2 instead of two material points. The only difference is that the configuration space is transformed by a translation:

$$\left\{(y_1, y_2):\ \frac{y_1}{\sqrt{m_1}} - \frac{y_2}{\sqrt{m_2}} + r_1 + r_2 \le 0\right\}.$$

Therefore, systems of two hard balls on a straight line are isomorphic to billiards in a half-plane.

Now consider a system of $n+1$ elastic material points on a line. Let $x_0(t), x_1(t), \dots, x_n(t)$ be the coordinates of the points satisfying

$$x_0(t) \le x_1(t) \le \cdots \le x_n(t), \quad \forall t.$$

And let $m_0, m_1, \dots, m_n$ be the corresponding masses. Take a scaling:

$$y_i(t) = \sqrt{m_i}\, x_i(t), \quad i = 0, 1, \dots, n. \tag{2.5}$$

Differentiating on t yields

$$\dot{y}_i(t) = \sqrt{m_i}\, \dot{x}_i(t), \quad i = 0, 1, \dots, n.$$

In an interval of time without collision, any material point is in uniform rectilinear motion, that is, $\dot{x}_i$ is constant for any i, thus the system velocity vector in the new variables

$$\dot{y} = (\dot{y}_0(t), \dot{y}_1(t), \dots, \dot{y}_n(t))^T \tag{2.6}$$

is also constant in the interval of time. Now the configuration space is

$$\left\{y \in \mathbb{R}^{n+1}:\ \frac{y_i}{\sqrt{m_i}} - \frac{y_{i-1}}{\sqrt{m_{i-1}}} \ge 0,\ i = 1, \dots, n\right\}. \tag{2.7}$$

Suppose t_0 is an instant of a simple collision, say, between the $(i-1)$th and ith material points, then $x_{i-1}(t_0) = x_i(t_0)$, i.e., $\frac{y_i(t_0)}{\sqrt{m_i}} - \frac{y_{i-1}(t_0)}{\sqrt{m_{i-1}}} = 0$. And we have

$$\dot{y}_j(t_0-) = \dot{y}_j(t_0+), \quad \forall j \ne i-1, i. \tag{2.8}$$

From the case of systems of two material points, we know

$$\begin{pmatrix} \dot{y}_{i-1}(t_0+) \\ \dot{y}_i(t_0+) \end{pmatrix} = \begin{pmatrix} \dot{y}_{i-1}(t_0-) \\ \dot{y}_i(t_0-) \end{pmatrix} - \frac{2}{\frac{1}{m_{i-1}} + \frac{1}{m_i}} \left\langle \begin{pmatrix} \dot{y}_{i-1}(t_0-) \\ \dot{y}_i(t_0-) \end{pmatrix}, \begin{pmatrix} -\frac{1}{\sqrt{m_{i-1}}} \\ \frac{1}{\sqrt{m_i}} \end{pmatrix} \right\rangle \begin{pmatrix} -\frac{1}{\sqrt{m_{i-1}}} \\ \frac{1}{\sqrt{m_i}} \end{pmatrix}.$$

Combining with (2.8) yields

$$\dot{y}(t_0+) = \dot{y}(t_0-) - 2\langle \dot{y}(t_0-), h_i\rangle h_i = \sigma_i(\dot{y}(t_0-)), \tag{2.9}$$

where

$$h_i = \frac{1}{\sqrt{\frac{1}{m_{i-1}} + \frac{1}{m_i}}} \left(0, \ldots, 0, -\frac{1}{\sqrt{m_{i-1}}}, \frac{1}{\sqrt{m_i}}, 0, \ldots, 0\right)^T \tag{2.10}$$

is the unit inner normal vector of the configuration space at the ith wall W_i which is included in the hyperplane

$$H_i = \left\{ y \in \mathbb{R}^{n+1} : \frac{y_i}{\sqrt{m_i}} - \frac{y_{i-1}}{\sqrt{m_{i-1}}} = 0 \right\}, \tag{2.11}$$

and σ_i is the orthogonal reflection about the hyperplane H_i. The reflection σ_i can be represented as

$$\sigma_i = I - 2h_i h_i^T, \quad 1 \le i \le n, \tag{2.12}$$

where I is the $(n+1) \times (n+1)$ identity matrix.

If we consider hard balls with radii $r_0, r_1, \ldots, r_n$ successively instead of material points, then the configuration space is

$$\begin{aligned} &\left\{ y \in \mathbb{R}^{n+1} : \frac{y_i}{\sqrt{m_i}} - \frac{y_{i-1}}{\sqrt{m_{i-1}}} \ge r_i + r_{i-1},\ i = 1, \ldots, n \right\} \\ &= \left\{ y \in \mathbb{R}^{n+1} : \langle y, h_i \rangle \ge \frac{r_i + r_{i-1}}{\sqrt{\frac{1}{m_i} + \frac{1}{m_{i-1}}}},\ i = 1, \ldots, n \right\}. \end{aligned} \tag{2.13}$$

Note that the unit normal vectors $h_1, \ldots, h_n$ are linearly independent, hence the configuration space is a simple polyhedral cone in $\mathbb{R}^{n+1}$ with n walls.

In sum, we have proved arbitrary system of $n+1$ hard balls on a line is isomorphic to a billiard inside a simple polyhedral cone in $\mathbb{R}^{n+1}$ with n walls. Furthermore, the n unit inner normal vectors $h_1, \ldots, h_n$ of the walls satisfy $\langle h_i, h_{i+1} \rangle < 0$ for each i and $\langle h_i, h_j \rangle = 0$ if $|i - j| > 1$. That is, all the dihedral angles between the walls W_i and W_{i+1} are acute and W_i is perpendicular to W_j if the indexes i and j are not successive. The following formula is significant for our main results, for $i = 1, \ldots, n-1$,

$$\langle h_i, h_{i+1} \rangle = -\frac{1}{m_i} \cdot \frac{1}{\sqrt{\frac{1}{m_{i-1}} + \frac{1}{m_i}}} \cdot \frac{1}{\sqrt{\frac{1}{m_i} + \frac{1}{m_{i+1}}}} = -\left[\left(1 + \frac{m_i}{m_{i-1}}\right)\left(1 + \frac{m_i}{m_{i+1}}\right)\right]^{-\frac{1}{2}}. \tag{2.14}$$

The billiard in $\mathbb{R}^{n+1}$ with n walls can be reduced into the subspace $\operatorname{span}\{h_1, \ldots, h_n\}$. From (2.9), we can see that the component of the velocity of the point particle in the direction of orthogonal complement of the space spanned by $h_1, \ldots, h_n$ is invariant. Thus the motion is a sum of two parts: billiard motion in the space spanned by $h_1, \ldots, h_n$ and uniform rectilinear motion in the orthogonal complement. That is, the configuration space (2.13) is a direct product

$$\mathbb{R} \times \left\{ y \in \operatorname{span}\{h_1, \ldots, h_n\} : \langle y, h_i \rangle \ge \frac{r_i + r_{i-1}}{\sqrt{\frac{1}{m_i} + \frac{1}{m_{i-1}}}},\ i = 1, \ldots, n \right\}, \tag{2.15}$$

the projection of the dynamics in the first factor is trivial, and the projection in the second factor is still a billiard. Note that the orthogonal complement of $\operatorname{span}\{h_1, \ldots, h_n\}$ is spanned by the vector

$$\mathbf{1} := \left(\sqrt{m_0}, \sqrt{m_1}, \dots, \sqrt{m_n}\right)^T \tag{2.16}$$

and the momentum is given by

$$M = \langle \mathbf{1}, \dot{y} \rangle. \tag{2.17}$$

Furthermore,

$$\langle \mathbf{1}, y(t) \rangle = \sum_{i=0}^{n} m_i x_i(t) \tag{2.18}$$

is the position function of the center of mass. Hence the reduction above corresponds to in the original hard ball system using the inertial barycentric reference system in which the center of mass is fixed at the origin according to the momentum conservation law.

The argument transforming any hard ball system on a line to a billiard inside a simple polyhedral cone is also valid for that on a half-line which is the limit case $m_0 = +\infty$, $y_0 \equiv 0$. Note that

$$\frac{\mathbf{1}}{\|\mathbf{1}\|} = \frac{\mathbf{1}}{\sqrt{\sum_{i=0}^{n} m_i}} \longrightarrow \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathbb{R}^{n+1}, \quad \text{when } m_0 \to +\infty,$$

and the orthogonal complement of $\frac{\mathbf{1}}{\|\mathbf{1}\|}$ is spanned by $h_1, \dots, h_n$. The configuration space is

$$\left\{ y \in \mathbb{R}^{n+1} : \; y_0 = 0, \; \langle y, h_i \rangle \ge \frac{r_i + r_{i-1}}{\sqrt{\frac{1}{m_i} + \frac{1}{m_{i-1}}}}, \; i = 1, \dots, n \right\}, \tag{2.19}$$

where $m_0 = +\infty$ and h_i is still given by (2.10). Thus the billiard is indeed inside a simple polyhedral cone in $\mathbb{R}^n$ with n walls.

In general, consider a hard ball system over a Riemannian manifold $\mathcal{M}$ with boundary $\partial\mathcal{M}$ constituted by several smooth components. Suppose there is N particles over $\mathcal{M}$ with masses $m_i > 0$ and potential radii $r_i \ge 0$ respectively, $1 \le i \le N$. Let g be the metric of $\mathcal{M}$, and ρ the distance function induced from g. Then the configuration space is

$$\mathcal{M}^N \Big\backslash \left(\bigcup_{i \neq j} C_{ij} \cup \bigcup_{i} D_i \right) \tag{2.20}$$

where

$$C_{ij} = \left\{ (x_1, \dots, x_N) \in \mathcal{M}^N : \; \rho(x_i, x_j) < r_i + r_j \right\}, \quad 1 \le i < j \le N, \tag{2.21}$$

and

$$D_i = \left\{ (x_1, \dots, x_N) \in \mathcal{M}^N : \; \rho(x_i, \partial\mathcal{M}) < r_i \right\}, \quad 1 \le i \le N. \tag{2.22}$$

The metric of the configuration space is given by

$$\tilde{g}\left((X_1, \dots, X_N), \left(X'_1, \dots, X'_N\right) \right) = \sum_{j=1}^{N} m_j \, g\left(X_j, X'_j\right). \tag{2.23}$$

Thus the hard ball system over $\mathcal{M}$ is isomorphic to the billiard over the configuration space. If $\mathcal{M}$ is the Euclidean space, take a scaling $y_i = \sqrt{m_i}x_i$, $1 \le i \le N$, then the metric of the configuration space is given by the standard scalar product in the new variables. Similarly, if $\mathcal{M}$ is a torus, since the coordinate system has a global expression, the metric can be treated in the same manner.

At the end of this section, let us say a few words on the interesting case $\mathcal{M}$ is a circle. In this case, the configuration space is a polyhedron. More precisely, if there is n particles on the circle, then the configuration space is a direct product of $\mathbb{R}$ and an $(n-1)$-dimensional simplex. The billiard possesses of a reduction into the simplex according to the conservation law of momentum. In particular, when $n = 3$, the billiard is reduced into an acute triangular. It is important that the formula (2.14) is still valid if we identify $m_{i+n} = m_i$, in particular $m_0 = m_n$! That is, the successive acute dihedral angles of the $(n-1)$-dimensional simplex are given by

$$\begin{aligned}\theta_i &= \pi - \arccos\langle h_i, h_{i+1}\rangle = \arccos(-\langle h_i, h_{i+1}\rangle)\\ &= \arccos\left[\left(1 + \frac{m_i}{m_{i-1}}\right)\left(1 + \frac{m_i}{m_{i+1}}\right)\right]^{-\frac{1}{2}}, \quad i = 1, \dots, n. \qquad (2.24)\end{aligned}$$

And all the other dihedral angles are $\frac{\pi}{2}$. The limit case $m_0 = m_n = +\infty$ describes a hard ball system in an interval and the formula still applies, with θ_n becoming a right angle.

3 A Discussion on Multiple Collisions

A collision between more than two objects is called a multiple collision, for instances, a collision between three or more hard balls, the collision when the billiard particle hits a corner.

The isomorphism established in the preceding section sheds some light on questions of multiple collisions. As a consequence, we see that questions on multiple collisions in a hard ball system are equivalent to those in the corresponding billiard in which the difficulty is revealed geometrically.

A multiple collision of the $(i-1)$th, ith, $\dots$, jth hard balls on a line corresponds to the billiard particle is meeting the corner constituted exactly by the walls $W_i, W_{i+1}, \dots, W_j$. In general, it is not appropriate to uniquely extend the trajectory after the collision. An event, however, that the billiard particle hits a corner whose walls are pairwise perpendicular does not mean a multiple collision in the original hard ball system, it is just several simultaneous simple collisions. In this case, the corresponding billiard trajectory of course should be extended as like the billiard particle meets the walls of the corner consecutively. In the geometrical aspect, it works since the orthogonal reflections about the walls commute with each other.

There are other cases in which it seems reasonable to extend the billiard trajectory after a multiple collision. The idea is to resolve the multiple collision into a sequence of consecutive simple collisions at the cone, and to determine when all the possible sequences of consecutive simple collisions give the same extension which we will refer to local regularization. We have already seen the case of pairwise perpendicular walls.

It turns out that a trajectory is locally regularizable at a cone if and only if, for any pair of two walls of the cone, there is a natural number k greater than 1 such that the dihedral angle between them is $\frac{\pi}{k}$, and consequently one trajectory is locally regularizable at the cone implies everyone does. Thus in this case we call the configuration space locally regular at the cone, and call the dynamical system locally regularizable if the configuration space is locally regular everywhere.

We use the phrase "locally regularizable" to distinguish it from "regularizable" in [4]. The configuration space is regular if and only if it is a fundamental domain of a finite Coxeter group, that is far more restrictive than local regularity. (Of course, if the configuration space is just a cone, regularity is equivalent to local regularity.) So being regularizable in fact implies being integrable.

Let us figure it out what means for a 1-dimensional hard ball system being (locally) regularizable. Consider the resulting simple polyhedral cone, besides the right dihedral angles, the angle between a pair of successive walls, say W_i and W_{i+1}, is equal to

$$\arccos\left[\left(1+\frac{m_i}{m_{i-1}}\right)\left(1+\frac{m_i}{m_{i+1}}\right)\right]^{-\frac{1}{2}}.$$

Hence the simple polyhedral cone is (locally) regular if and only if for each i, $1 \le i \le n-1$, there is an integer $k \ge 2$ such that

$$\left(1+\frac{m_i}{m_{i-1}}\right)\left(1+\frac{m_i}{m_{i+1}}\right) = \cos^{-2}\frac{\pi}{k}. \tag{3.1}$$

To the author of the paper, regularization of the model is misleading in some sense. And most of authors just leave the multiple collisions in a mathematical billiard undefined, analogous to that a measurable function needs not to be defined on a domain of zero measure. In the author's opinion, it is more natural to think the dynamics at multiple collisions are set-valued, analogous to think an interval exchange transformation is set-valued (in fact, two-valued) at points of discontinuity. A multiple collision is essentially a set of different sequences of consecutive simple collisions accumulated into one event by the model. And by definition, the dynamics are always single-valued if and only if the configuration space is locally regular.

The set-valued view of point is not a kind of trivial definitions. It might increase our real knowledge even in the well studied area of billiards in polygons. Especially, it indicates to regarding billiard transformations (also called billiard ball maps) as higher-dimensional generalizations of interval exchange transformations. This seems to shed new light on questions of periodic billiard trajectories in obtuse triangles and irrational polygons, see for instance Chap. 7 of [8] and references therein. On the other hand, the set-valued point view justifies us to remove the artificial assumption that multiple collisions do not occur for estimates of the number of collisions.

4 Using Inversion Numbers

Let P be an (unbounded) polyhedron in $\mathbb{R}^m$ with walls $W_1, \dots, W_n$. Denote h_i the unit inner normal vector of W_i for each i. Suppose W_i is perpendicular to W_j if $|i-j|>1$, and the angle between W_i and W_{i+1} is not less than $\frac{\pi}{3}$ for $1 \le i \le n-1$. That is,

$$\begin{cases} \langle h_i, h_j\rangle = 0, & |i-j|>1; \\ \langle h_i, h_{i+1}\rangle \ge -\frac{1}{2}, & 1 \le i \le n-1. \end{cases} \tag{4.1}$$

Recall that the inversion number of a sequence of numbers $\mathbf{q} = (q_0, q_1, \dots, q_n)$, denoted by $\operatorname{inv}\mathbf{q}$, is the number of inversions $(i,j): i<j, q_i>q_j$. It is obvious $0 \le \operatorname{inv}\mathbf{q} \le \frac{n(n+1)}{2}$.

Theorem 4.1 *Let T be a billiard trajectory in the polyhedron P. The velocity of T is denoted by $v(t)$. Set $q_0(t) \equiv 0$,*

$$q_i(t) = \sum_{j=1}^{i} \langle v(t), h_j \rangle, \tag{4.2}$$

and

$$\mathbf{q}(t) = (q_0(t), q_1(t), \ldots, q_n(t)). \tag{4.3}$$

Suppose t_0 is not an instant of a collision, then the number of collisions of T after t_0 is not bigger than $\operatorname{inv} \mathbf{q}(t_0)$. *Hence, for any billiard trajectory in P, the number of collisions is not bigger than $\frac{n(n+1)}{2}$.*

Proof Assume t_1 is an instant of a collision of the trajectory at the wall W_i. Then

$$\langle v(t_1-), h_i \rangle < 0, \quad \text{i.e., } q_i(t_1-) < q_{i-1}(t_1-). \tag{4.4}$$

To prove the theorem, it suffices to show

$$\operatorname{inv} \mathbf{q}(t_1+) < \operatorname{inv} \mathbf{q}(t_1-). \tag{4.5}$$

When $\operatorname{inv} \mathbf{q}(t) = 0$, that is, $\mathbf{q}(t)$ is an increasing sequence, we have $\langle v(t), h_i \rangle \geq 0$ for each i, hence there are no collision any more.

For each j, clearly we have

$$\langle v(t_1+), h_j \rangle = \langle v(t_1-), h_j \rangle - 2\langle v(t_1-), h_i \rangle \langle h_i, h_j \rangle.$$

When $|j - i| > 1$, $\langle h_i, h_j \rangle = 0$ implies $\langle v(t_1+), h_j \rangle = \langle v(t_1-), h_j \rangle$. It immediately follows

$$q_j(t_1+) = q_j(t_1-), \quad j \leq i - 2.$$

And it turns out

$$\begin{aligned} q_{i-1}(t_1+) &= q_{i-2}(t_1+) + \langle v(t_1+), h_{i-1} \rangle \\ &= q_{i-2}(t_1-) + \langle v(t_1-), h_{i-1} \rangle - 2\langle v(t_1-), h_i \rangle \langle h_i, h_{i-1} \rangle \\ &= q_{i-1}(t_1-) + \langle v(t_1-), h_i \rangle - \langle v(t_1-), h_i \rangle - 2\langle v(t_1-), h_i \rangle \langle h_i, h_{i-1} \rangle \\ &= q_i(t_1-) - \langle v(t_1-), h_i \rangle (1 + 2\langle h_i, h_{i-1} \rangle); \end{aligned}$$

$$\begin{aligned} q_i(t_1+) &= q_{i-1}(t_1+) + \langle v(t_1+), h_i \rangle \\ &= q_i(t_1-) - \langle v(t_1-), h_i \rangle (1 + 2\langle h_i, h_{i-1} \rangle) - \langle v(t_1-), h_i \rangle \\ &= q_{i-1}(t_1-) - \langle v(t_1-), h_i \rangle (1 + 2\langle h_i, h_{i-1} \rangle); \end{aligned}$$

$$\begin{aligned} q_{i+1}(t_1+) &= q_i(t_1+) + \langle v(t_1+), h_{i+1} \rangle \\ &= q_{i-1}(t_1-) - \langle v(t_1-), h_i \rangle (1 + 2\langle h_i, h_{i-1} \rangle) \\ &\quad + \langle v(t_1-), h_{i+1} \rangle - 2\langle v(t_1-), h_i \rangle \langle h_i, h_{i+1} \rangle \\ &= q_{i+1}(t_1-) - \langle v(t_1-), h_i \rangle (1 + 2\langle h_i, h_{i-1} \rangle + 1 + 2\langle h_i, h_{i+1} \rangle). \end{aligned}$$

The rest computation is easy,

$$q_j(t_1+) = q_j(t_1-) - \langle v(t_1-), h_i\rangle(1 + 2\langle h_i, h_{i-1}\rangle + 1 + 2\langle h_i, h_{i+1}\rangle), \quad j \geq i+2.$$

Let $\mathbf{k} = (k_0, k_1, \ldots, k_n)$ where

$$k_j = \begin{cases} 0, & j \leq i-2; \\ 1 + 2\langle h_i, h_{i-1}\rangle, & j = i-1 \text{ or } i; \\ 1 + 2\langle h_i, h_{i-1}\rangle + 1 + 2\langle h_i, h_{i+1}\rangle, & j \geq i+1. \end{cases} \tag{4.6}$$

Then $\mathbf{k}$ is an increasing sequence by hypothesis. And let $\tilde{\mathbf{q}}(t_1+)$ be the sequence obtained from $\mathbf{q}(t_1-)$ by interchanging the places of $q_{i-1}(t_1-)$ and $q_i(t_1-)$. Then

$$\mathbf{q}(t_1+) = \tilde{\mathbf{q}}(t_1+) - \langle v(t_1-), h_i\rangle \mathbf{k}. \tag{4.7}$$

From (4.4) we see

$$\operatorname{inv} \tilde{\mathbf{q}}(t_1+) = \operatorname{inv} \mathbf{q}(t_1-) - 1 \tag{4.8}$$

and $-\langle v(t_1-), h_i\rangle \mathbf{k}$ is an increasing sequence. Hence $\operatorname{inv} \mathbf{q}(t_1+) \leq \operatorname{inv} \mathbf{q}(t_1-) - 1$, which completes the proof. □

Now we turn to apply the preceding theorem to hard ball systems on a line using the results of Sect. 2. Let $m_0, m_1, \ldots, m_n$ be the masses successively and $x_0, x_1, \ldots, x_n$ the corresponding coordinates as before. The first corollary applies not only to systems on a whole-line, but also to those on a half-line: set $m_0 = +\infty$ (or $m_n = +\infty$). The other corollaries all follow from the first.

Corollary 4.2 *If* $(1 + \frac{m_i}{m_{i-1}})(1 + \frac{m_i}{m_{i+1}}) \geq 4$ *for each* i, $1 \leq i \leq n-1$, *then the number of collisions in the hard ball system is not bigger than* $\frac{n(n+1)}{2}$. *Moreover, denote the sequence* $\mathbf{q}(t) = (q_0(t), q_1(t), \ldots, q_n(t))$, *where for each* i,

$$q_i(t) = \sum_{j=1}^{i} \frac{\dot{x}_j(t) - \dot{x}_{j-1}(t)}{\sqrt{\frac{1}{m_j} + \frac{1}{m_{j-1}}}} \quad (q_0(t) \equiv 0), \tag{4.9}$$

then the number of collisions in the system after a certain instant t *is not bigger than the inversion number* $\operatorname{inv} \mathbf{q}(t)$.

Proof It is because here $\langle h_i, h_{i+1}\rangle \geq -\frac{1}{2}$ is equivalent to $(1 + \frac{m_i}{m_{i-1}})(1 + \frac{m_i}{m_{i+1}}) \geq 4$. The involved computations directly follow from the formulas (2.5), (2.6), (2.10) and (2.14). □

Corollary 4.3 [2] *If* $m_i \geq \sqrt{m_{i-1}m_{i+1}}$ *for each* i, *then the number of collisions is not bigger than* $\frac{n(n+1)}{2}$.

Proof

$$\left(1 + \frac{m_i}{m_{i-1}}\right)\left(1 + \frac{m_i}{m_{i+1}}\right) \geq \left(1 + \sqrt{\frac{m_{i+1}}{m_{i-1}}}\right)\left(1 + \sqrt{\frac{m_{i-1}}{m_{i+1}}}\right)$$
$$= 2 + \sqrt{\frac{m_{i+1}}{m_{i-1}}} + \sqrt{\frac{m_{i-1}}{m_{i+1}}} \geq 2 + 2 = 4.$$

□

The next is the (right) half-line version of the preceding corollary.

Corollary 4.4 *If* $m_0 = +\infty, m_1 \geq 3m_2$, *and* $m_i \geq \sqrt{m_{i-1}m_{i+1}}$ *when* $2 \leq i \leq n-1$. *Then the number of collisions does not succeed* $\frac{n(n+1)}{2}$.

Proof $(1 + \frac{m_1}{m_0})(1 + \frac{m_1}{m_2}) = 1 + \frac{m_1}{m_2} \geq 1 + 3 = 4.$ □

If the condition $m_1 \geq 3m_2$ is replaced by $m_1 \geq m_2$, we still have an upper bound of second order.

Corollary 4.5 *If* $m_0 = +\infty, m_1 \geq m_2$, *and* $m_i \geq \sqrt{m_{i-1}m_{i+1}}$ *when* $2 \leq i \leq n-1$. *Then the number of collisions does not succeed* $n(2n-1)$.

Proof Let us reflect symmetrically the whole dynamical system at the end of the half-line. We obtain a system of $2n$ hard balls on a line moving symmetrically with a wall in the middle. After a collision of the two neighboring hard balls with the wall, the two balls of identical mass move with equal and opposite velocities as like that they have collided with each other without the wall, having interchanged the velocities. Therefore, we can remove the wall away without changing the dynamics. Then it follows from Corollary 4.3 that the number of collisions does not succeed $\frac{(2n-1)\cdot 2n}{2} = n(2n-1)$. □

Corollary 4.6 *If* $\frac{m_{2k+1}}{m_{2k}} = \frac{k}{k+1}$ $(k \geq 0)$ *and* $\frac{m_{2k}}{m_{2k-1}} = \frac{2k-1}{2k+1}$ $(k \geq 1)$. *Then the number of collisions is not bigger than* $\frac{n(n+1)}{2}$.

Proof If $i = 2k$ $(k \geq 1)$, then

$$\left(1 + \frac{m_i}{m_{i-1}}\right)\left(1 + \frac{m_i}{m_{i+1}}\right) = \left(1 + \frac{2k-1}{2k+1}\right)\left(1 + \frac{k+1}{k}\right) = \frac{4k}{2k+1} \cdot \frac{2k+1}{k} = 4;$$

if $i = 2k+1$ $(k \geq 0)$, then

$$\left(1 + \frac{m_i}{m_{i-1}}\right)\left(1 + \frac{m_i}{m_{i+1}}\right) = \left(1 + \frac{k}{k+1}\right)\left(1 + \frac{2k+3}{2k+1}\right) = \frac{2k+1}{k+1} \cdot \frac{4k+4}{2k+1} = 4.$$ □

We state the last corollary since the involved number sequence $\{a_k\}_{k=0}^{+\infty}$ is interesting, where $a_{2k} = \frac{k}{k+1}$ $(k \geq 0)$ and $a_{2k-1} = \frac{2k-1}{2k+1}$ $(k \geq 1)$. The sequence starts as follows:

$$\frac{0}{1}, \frac{1}{3}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{5}{7}, \frac{3}{4}, \frac{7}{9}, \frac{4}{5}, \frac{9}{11}, \frac{5}{6}, \ldots. \tag{4.10}$$

Let $a_n = \frac{p_n}{q_n}$ in the reduced form. The sequence satisfies the following properties: (1) $a_n + \frac{1}{a_{n+2}} = 2$; (2) $q_n p_{n+1} - p_n q_{n+1} = 1$; (3) $\frac{p_{n-1}+p_{n+1}}{q_{n-1}+q_{n+1}} = \frac{p_n}{q_n}$. And one can find this sequence near 1 in the map of relatively prime pairs which displays all the positive rational numbers in the reduced form by a complete binary tree, see the picture in the cover of [7]!

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